

1. Using Taylor Tables

(a) the finite difference scheme for the 3rd derivative

$$\delta_{xxx}u_j = \frac{-u_{j-2} + bu_{j-1} + cu_j + du_{j+1} + u_{j+2}}{a\Delta x^3}$$

has

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5 \cdot \left(\frac{\partial^5 u}{\partial x^5}\right)_j$
$a\Delta x^3 \cdot \delta_{xxx}u_j$	0	0	0	a	0	0
u_{j-2}	1	$1(-2)^1 \frac{1}{1!}$	$1(-2)^2 \frac{1}{2!}$	$1(-2)^3 \frac{1}{3!}$	$1(-2)^4 \frac{1}{4!}$	$1(-2)^5 \frac{1}{5!}$
$-bu_{j-1}$	$-b$	$-b(-1)^1 \frac{1}{1!}$	$-b(-1)^2 \frac{1}{2!}$	$-b(-1)^3 \frac{1}{3!}$	$-b(-1)^4 \frac{1}{4!}$	$-b(-1)^5 \frac{1}{5!}$
$-cu_j$	$-c$	0	0	0	0	0
$-du_{j+1}$	$-d$	$-d(1)^1 \frac{1}{1!}$	$-d(1)^2 \frac{1}{2!}$	$-d(1)^3 \frac{1}{3!}$	$-d(1)^4 \frac{1}{4!}$	$-d(1)^5 \frac{1}{5!}$
$-u_{j+2}$	-1	$-1(2)^1 \frac{1}{1!}$	$-1(2)^2 \frac{1}{2!}$	$-1(2)^3 \frac{1}{3!}$	$-1(2)^4 \frac{1}{4!}$	$-1(2)^5 \frac{1}{5!}$
	0	0	0	0	?	?

where the first 4 columns are set to 0, resulting in the four equations for the four unknowns

$$b + c + d = 0$$

$$b - d = 4$$

$$b + d = 0$$

$$6a + b - d = 16$$

You can either solve them directly or put them into matrix form and solve, resulting in $[a, b, c, d] = [2, 2, 0, -2]$

(b) Actually to prove 2nd order accuracy one needs to eliminate the fifth column, which is $b+d=0$, so it is satisfied. Evaluating the sixth column results in

$$er_t = \frac{1}{a\Delta x^3 \left(\frac{b-d-64}{5!}\right)} \Delta x^5 \left(\left(\frac{\partial^5 u}{\partial x^5}\right)_j\right) = -\frac{1}{4} \Delta x^2 \left(\left(\frac{\partial^5 u}{\partial x^5}\right)_j\right)$$

which is second order accurate.

(c) For the continuous function $u(x) = e^{ikx}$ the third derivative gives $\partial_{xxx}e^{ikx} = -ik^3e^{ikx}$. Using the discrete function $u_j = e^{ikj\Delta x}$ produces $\delta_{xxx}u_j = -i(k^*)^3u_j$. Applying this to the difference equation leads to

$$\begin{aligned} -i(k^*)^3 &= \frac{1}{2\Delta x^3} \left[-e^{-2ik\Delta x} + 2e^{-ik\Delta x} - 2e^{ik\Delta x} + e^{2ik\Delta x} \right] \\ &= \frac{i}{\Delta x^3} [\sin 2k\Delta x - 2\sin k\Delta x] \end{aligned}$$

This gives

$$(k^*)^3 = \frac{2\sin k\Delta x - \sin 2k\Delta x}{\Delta x^3}$$

(d) Expanding using the series expression for sin and reducing leads to

$$(k^*)^3 = k^3 - \frac{1}{4}k^5\Delta x^2 + \dots$$

which confirms the second order accuracy from 1b. Note: I did not try to reduce $(k^*)^3$ by taking the third root and result 1d serves as a check for 1b and visa versa.

2. Applying the representative equation $u' = \lambda u$ to the OΔE scheme

$$u_{n+1} = u_{n-1} + h(2\beta_1(u')_n + \beta_0(u')_{n-1})$$

results in

(a)

$$\begin{aligned} Eu_n &= E^{-1}u_n + 2\beta_1 h \lambda u_n + \beta_0 h \lambda E^{-1}u_n \\ P(E) &= E - 2\beta_1 h \lambda - (1 + \beta_0 h \lambda) E^{-1} \\ P(\sigma) &= 0 \rightarrow \sigma_{1,2} = h \lambda \beta_1 \pm \sqrt{1 + \beta_0 h \lambda + h^2 \lambda^2 \beta_1^2} \end{aligned}$$

(b) Principal and spurious roots can be identified by replacing $h \lambda = 0$ giving us $\sigma_1 = 0 + \sqrt{1} = 1$ (the principal root) and $\sigma_2 = 0 - \sqrt{1} = -1$ (the spurious root).

(c) Taking σ_1 , expanding the square root term and forming er_t we get

$$er_t = e^{\lambda h} - h \lambda \beta_1 - 1 - \frac{1}{2}(\beta_0 h \lambda + \beta_1^2 h^2 \lambda^2) + \frac{1}{8}(\beta_0 h \lambda + \beta_1^2 h^2 \lambda^2)^2 + \dots$$

- i. For 1st Order Accuracy, we have the condition $\beta_0 + 2\beta_1 = 2$, which gives $er_t = O(h^2)$.
- ii. For 2nd Order Accuracy, we have the condition $\beta_0 = 0, \beta_1 = 1$, which gives $er_t = O(h^3)$.

3. Applying the representative equation to the predictor-corrector scheme

$$\begin{aligned} \bar{u}_{n+1} &= u_n + h(u')_n \\ u_{n+1} &= u_n + \frac{1}{2}h(3(\bar{u}')_{n+1} - (u')_n) \end{aligned}$$

results in the matrix form

(a)

$$\begin{bmatrix} E & -(1 + h\lambda) \\ -\frac{3}{2}h\lambda E & E - 1 + \frac{1}{2}h\lambda \end{bmatrix} \begin{bmatrix} \bar{u}^n \\ u^n \end{bmatrix} = \begin{bmatrix} \frac{3}{2}hE - \frac{1}{2}h \\ \frac{3}{2}hE - \frac{1}{2}h \end{bmatrix} a e^{\mu h n}$$

giving us

$$[P(E)]\vec{u}^n = [\vec{Q}(E)]a e^{\mu h n}$$

The characteristic polynomial $P(E)$ is obtained as $P(E) = \text{determinant of } [P(E)]$ giving

$$P(E) = E[E - 1 - h\lambda - \frac{3}{2}h\lambda^2]$$

(b) The σ roots are obtained by letting $P(\sigma) = 0$ which gives us the trivial root $\sigma_2 = 0$ and

$$\sigma_1 = 1 + h\lambda + \frac{3}{2}(h\lambda)^2$$

(c) Using the series expansion of

$$e^{\lambda h} = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \dots$$

we have $er_\lambda = -h^2 \lambda^2$ showing a 1st order method. *Note: common mistake is to say this is a 2nd Order method. Take off one power of h for the order of accuracy for to u_n .*

4. EXTRA CREDIT PROBLEMS

A system of PDE's produces a $\lambda = \alpha + i\beta$

- (a) The resulting ODE is stable for $\alpha > 0$.
 - i. This is **FALSE**, the $Re(\lambda) \leq 0$ for inherent stability of ODE.
- (b) An ODE with $\sigma = 1 + \lambda h$ and $\alpha = 0, \beta \neq 0$ is unconditionally unstable.
 - i. This is **TRUE**, $|\sigma| = \sqrt{1 + \beta^2 h^2} \geq 1$ for all βh , leading to unconditionally instability.

Independent of λ

- (c) The Leapfrog Scheme produces the two roots:

$$\begin{aligned}\sigma_1 &= \lambda h + \sqrt{1 + (\lambda h)^2} \\ \sigma_2 &= \lambda h - \sqrt{1 + (\lambda h)^2}\end{aligned}$$

where σ_2 is the principal root.

- i. This is **FALSE**, $\sigma_1 \rightarrow 1$ as $\lambda h \rightarrow 0$ and $\sigma_2 \rightarrow -1$, showing that σ_1 is the principal root and σ_2 is the spurious root.

NOTE: I only gave extra points if you gave a valid explanation, not just for the T/F result.